Finite Basis Problems and Results for Quasivarieties

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Problem

All algebras are assumed to be of finite signature.

Definition. A class $\mathcal{K}$ of algebras is *finitely axiomatizable* if there exists a first-order sentence ϕ such that $\mathcal{K} = \mathrm{Mod}(\phi)$. For varieties (or quasivarieties) this condition is equivalent to having a *finite basis of equations* (or *quasiequations*, respectively).

Problem. *Which varieties (or quasivarieties) are finitely axiomatizable?*

Problem. *Under what conditions can one decide of a finite algebra if it generates a finitely axiomatizable variety (or quasivariety)?*

Classical Results

Theorem (R. Lyndon, 1951). *Every variety generated by a two-element algebra is finitely axiomatizable,*

Theorem (S. Oates, M. B. Powell, 1964). *Every finitely generated variety of groups is finitely axiomatizable.*

Theorem (P. Perkins, 1969). *Every finitely generated variety of commutative semigroups is finitely axiomatizable.*

Theorem (J. A. Gerhard, 1970). *Every finitely generated variety of idempotent semigroups is finitely axiomatizable.*

Theorem (R. L. Kruse, I. V. L'vov, 1973). *Every finitely generated variety of associative rings is finitely axiomatizable.*

Nonfinitely Axiomatizable Examples

Examples of nonfinitely axiomatizable varieties generated by:

Example (R. Lyndon, 1951)**.** *a 7-element algebra.*

Example (V. V. Visin, 1963)**.** *a 4-element groupoid.*

Example (V. L. Murskiĭ, 1965)**.** *a 3-element groupoid.*

Example (P. Perkins, 1969)**.** *a 6-element semigroup.*

Example (S. V. Polin, 1976)**.** *a finite non-associative ring.*

Example (R. Bryant, 1982)**.** *a finite pointed group.*

Examples of nonfinitely axiomatizable quasivarieties generated by:

Example (V. P. Belkin, 1978)**.** *a 10-element lattice.*

Example (J. Lawrence, R. Willard, 1998)**.** *an 18-element semigroup with an additional unary operation (inherently nonfinitely based for quasi-equations).*

Universal Algebraic Results for Varieties

Theorem (K. Baker, 1977)**.** *Every finitely generated congruence-distributive variety is finitely axiomatizable.*

Theorem (R. McKenzie, 1987)**.** *Every finitely generated modular variety with a cardinal bound on the size of its subdirectly irreducible members is finitely axiomatizable.*

Definition. A lattice is *meet semi-distributive* if it satisfies

$$x \wedge y = x \wedge z \rightarrow x \wedge y = x \wedge (y \vee z).$$

Theorem (R. Willard, 2000)**.** *Every congruence meet semi-distributive variety with a finite bound on the size of its subdirectly irreducible members is finitely axiomatizable.*

Conjecture (R. E. Park, 1976)**.** *Every finitely generated variety with a finite bound on the size of its subdirectly irreducible members is finitely axiomatizable.*

Universal Algebraic Results for Quasivarieties

Definition. For a quasivariety $\mathcal{K}$ and algebra $\mathbf{A} \in \mathcal{K}$ the *relative congruence lattice of* $\mathbf{A}$ *with respect to* $\mathcal{K}$, denoted by $\operatorname{Con}_{\mathcal{K}} \mathbf{A}$, is the lattice-ordered set of congruences $\alpha \in \operatorname{Con} \mathbf{A}$ with $\mathbf{A}/\alpha \in \mathcal{K}$.

Theorem (D. Pigozzi, 1988)**.** *Every finitely generated relatively congruence distributive quasivariety is finitely axiomatizable.*

Conjecture (D. Pigozzi)**.** *Every finitely generated relatively modular quasivariety is finitely axiomatizable.*

Theorem (M.M., R. McKenzie, 2003)**.** *Let* $\mathcal{K}$ *be a finite set of finite algebras such that* $SP(\mathcal{K})$ *has pseudo-complemented congruence lattices. If* $HS(\mathcal{K}) \subseteq SP(\mathcal{K})$ *then* $SP(\mathcal{K})$ *is finitely axiomatizable.*

Pseudo-complemented Congruence Lattices

Definition. A lattice is *pseudo-complemented* if for every element a there exists a largest element b such that $a \wedge b = 0$. A quasivariety has *pseudo-complemented congruence lattices* if the congruence lattices of its members are pseudo-complemented.

Definition. A *set of Willard terms* for a quasivariety $\mathcal{K}$ is a finite sequence $\{ (f_i, g_i) : i < n \}$ of pairs of ternary terms such that the equations $f_i(x, y, x) \approx g_i(x, y, x)$ $(i < n)$ hold in $\mathcal{K}$ and so does

$$x \neq y \to \bigvee_{i<n} \Big(f_i(x, x, y) = g_i(x, x, y) \leftrightarrow f_i(x, y, y) \neq g_i(x, y, y) \Big).$$

Theorem. *For every quasivariety*

$$\text{CD} \Rightarrow \text{SD}(\wedge) \Rightarrow \text{PCC} \Leftrightarrow \text{W}.$$

Proof. (PCC)$\Leftarrow$(W): We will show that for all $\alpha, \beta, \gamma \in \operatorname{Con} \mathbf{A}$

$$\alpha \wedge \beta = \alpha \wedge \gamma = 0_{\mathbf{A}} \rightarrow \alpha \wedge (\beta \vee \gamma) = 0_{\mathbf{A}}.$$

Choose $(a, b) \in \alpha \wedge (\beta \vee \gamma) - 0_{\mathbf{A}}$. By (W) we have $i < n$ such that $f_i(a, a, b) = g_i(a, a, b)$ and $f_i(a, b, b) \neq g_i(a, b, b)$. Using a $\beta \vee \gamma$ Maltsev chain from a to b we can find $(c, d) \in \beta$ (or in γ) such that $f_i(a, c, b) = g_i(a, c, b)$ and $f_i(a, d, b) \neq g_i(a, d, b)$. Now

$$f_i(a, d, b) \, \alpha \, f_i(a, d, a) = g_i(a, d, a) \, \alpha \, g_i(a, d, b),$$

$$f_i(a, d, b) \, \beta \, f_i(a, c, b) = g_i(a, c, b) \, \beta \, g_i(a, d, b),$$

that is $f_i(a, d, b) \, (\alpha \wedge \beta) \, g_i(a, d, b)$, contradicting $\alpha \wedge \beta = 0_{\mathbf{A}}$. □

Theorem. *A locally finite quasivariety $\mathcal{K}$ has pseudo-complemented congruences iff no algebra in $\mathcal{K}$ has a non-trivial Abelian congruence.*

Proof. ($\Leftarrow$): Assume $\alpha \wedge \beta = \alpha \wedge \gamma = 0_{\mathbf{A}}$ but $\delta = \alpha \wedge (\beta \vee \gamma) > 0_{\mathbf{A}}$. Then β and γ centralize α, thus $\beta \vee \gamma$ centralizes α. Since $\delta \leq \alpha$ and $\delta \leq \beta \vee \gamma$, δ centralizes itself, i.e. δ is Abelian.

($\Rightarrow$): Assume (W), and suppose that there exists a minimal Abelian congruence δ in a finite algebra $\mathbf{A} \in \mathcal{K}$. Take $(a, b) \in \delta - 0_{\mathbf{A}}$ from a $(0_{\mathbf{A}}, \delta)$-trace N. By (W) we have $i < n$ so that $f_i(a, a, b) = g_i(a, a, b)$ while $f_i(a, b, b) \neq g_i(a, b, b)$. Recall that $f_i(x, y, x) = g_i(x, y, x)$ holds, as well. We can map $(f_i(a, b, b), g_i(a, b, b))$ by a non-collapsing idempotent unary polynomial back to N. Then by case analysis for types **1** and **2** we get a contradiction. □

Principal Congruence Disjointness

Definition. For an algebra $\mathbf{A}$ and integer m the *principal congruence m-disjointness relation over* $\mathbf{A}$ is the $2m$-ary relation

$$PCD_m(\bar{a};\bar{b}) \leftrightarrow \bigcap_{i<m} \theta_{\mathbf{A}}(a_i, b_i) = 0_{\mathbf{A}}.$$

Definition. Let ϕ be a positive universal sentence written as

$$\bigwedge_{i<k} (\forall x_0, \ldots, x_{n_i-1}) \bigvee_{j<m_i} \sigma_{i,j}(\bar{x}) \approx \tau_{i,j}(\bar{x})$$

where $\sigma_{i,j}, \tau_{i,j}$ are n_i-ary terms. By $PCD(\phi)$ we mean the condition

$$\bigwedge_{i<k} (\forall \bar{x}) PCD_{m_i}\big(\sigma_{i,0}(\bar{x}), \ldots, \sigma_{i,m_i-1}(\bar{x}); \tau_{i,0}(\bar{x}), \ldots, \tau_{i,m_i-1}(\bar{x})\big).$$

Theorem. *Let $\mathcal{K}$ be a quasivariety with pseudo-complemented congruence lattices, and ϕ be a positive universal sentence. Then $\mathcal{K} \cap SP(\mathrm{Mod}(\phi))$ is the class of algebras $\mathbf{A} \in \mathcal{K}$ satisfying $PCD(\phi)$.*

Proof. ($\Rightarrow$): Assume that $\mathbf{A} \leq \prod_{t \in T} \mathbf{A}_t$ with $\mathbf{A}_t \models \phi$ for all $t \in T$. For all choices of $i < k$, $\bar{a} \in A^{n_i}$ we need to show that

$$PCD_{m_i}\big(\sigma_{i,0}(\bar{x}), \ldots, \sigma_{i,m_i-1}(\bar{x}); \tau_{i,0}(\bar{x}), \ldots, \tau_{i,m_i-1}(\bar{x})\big).$$

This is equivalent to proving that $f = g$ for every pair

$$(f, g) \in \bigcap_{j<m_i} \theta_{\mathbf{A}}\big(\sigma_{i,j}(\bar{a}), \tau_{i,j}(\bar{a})\big).$$

Take any $t \in T$. Since $\mathbf{A}_t \models \phi$, there is $j < n_i$ such that $\sigma^{\mathbf{A}}_{i,j}(\bar{a})$ and $\tau^{\mathbf{A}}_{i,j}(\bar{a})$ agree at coordinate t, so $f(t) = g(t)$. This holds for all $t \in T$, therefore $f = g$. $\square$

Definition. For a class $\mathcal{K}$ of algebras and an integer n we define $\mathcal{K}_n$ to be the class of algebras having at most n elements.

Theorem. *Let $\mathcal{W}$ be a quasivariety with Willard terms, and n, m be integers. Then PCD_m is first-order definable in $\mathcal{W} \cap SP(H(\mathcal{W})_n)$ by a formula δ_m. Moreover, in all algebras of this type $PCD_m \to \delta_m$.*

A version of the this theorem was discovered by K. Baker, G. McNulty and Ju. Wang for congruence meet semi-distributive varieties with bounded critical depth.

Theorem. *Let* $\mathbf{A}$ *be an algebra with a pseudo-complemented congruence lattice and* $\delta(\bar{x};\bar{y})$ *be a* $2m$*-ary first-order formula for which* $\mathbf{A} \models PCD_m \to \delta$*. Then there exists a first-order sentence* $\gamma(\delta)$ *such that* $\mathbf{A} \models \gamma(\delta)$ *iff* PCD_m *is defined by* δ *over* $\mathbf{A}$*.*

Proof. ($\Leftarrow$) Take $\bar{a}, \bar{b} \in A^m$. Put $\alpha_0 = \bigcap_{0<i<m} \theta_{\mathbf{A}}(a_i, b_i)$ and let β_0 be the pseudo-complement of α_0. Clearly,

$$\beta_0 = \{\, (x, y) : \mathbf{A} \models \delta(x, a_1, \ldots, a_{m-1}; y, b_1, \ldots, b_{m-1}) \,\}.$$

Then put $\alpha_1 = \beta_0 \cap \bigcap_{1<i<m} \theta_{\mathbf{A}}(a_i, b_i)$, and let β_1 be the pseudo-complement of α_1. Now

$$\beta_1 = \{\, (x, y) : \mathbf{A} \models \delta(u, x, a_2, \ldots; v, y, b_2, \ldots) \text{ for all } (u, v) \in \beta_0 \,\}.$$

Recursively, we have first-order definable congruences $\beta_0, \ldots, \beta_m$ such that $\bigcap_{i<m} \beta_i = 0_{\mathbf{A}}$. We define $\gamma(\delta)$ to be the sentence that says

"the relations β_i are congruences and $\bigcap_{i<m} \beta_i = 0_{\mathbf{A}}$." □

Principal Congruence Disjointness Summary

Definition. For an algebra $\mathbf{A}$ and integer m the *principal congruence m-disjointness relation over* $\mathbf{A}$ is the $2m$-ary relation

$$PCD_m(\bar{a};\bar{b}) \leftrightarrow \bigcap_{i<m} \theta_{\mathbf{A}}(a_i, b_i) = 0_{\mathbf{A}}.$$

Theorem. *Let $\mathcal{K}$ be a quasivariety with pseudo-complemented congruence lattices, and ϕ be a positive universal sentence. Then $\mathcal{K} \cap SP(\mathrm{Mod}(\phi))$ is the class of algebras $\mathbf{A} \in \mathcal{K}$ that satisfy $PCD(\phi)$.*

Theorem. *Let $\mathcal{W}$ be a quasivariety with Willard terms, and n, m be integers. Then PCD_m is first-order definable in $\mathcal{W} \cap SP(H(\mathcal{W})_n)$ by a formula δ_m. Moreover, in all algebras of this type $PCD_m \rightarrow \delta_m$.*

Theorem. *Let $\mathbf{A}$ be an algebra with a pseudo-complemented congruence lattice and $\delta(\bar{x};\bar{y})$ be a $2m$-ary first-order formula for which $\mathbf{A} \models PCD_m \rightarrow \delta$. Then there exists a first-order sentence $\gamma(\delta)$ such that $\mathbf{A} \models \gamma(\delta)$ iff PCD_m is defined by δ over $\mathbf{A}$.*

Finite Basis Theorems for Quasivarieties

Theorem. *Let $\mathcal{W}$ be a quasivariety with pseudo-complemented congruence lattices, n be a positive integer, and ϕ be a positive universal sentence. Then $\mathcal{L} = \mathcal{W} \cap SP(H(\mathcal{W})_n) \cap SP(\mathrm{Mod}(\phi))$ is finitely axiomatizable relative to $\mathcal{W}$.*

Proof. By δ_m we mean the first-order sentence axiomatizing PCD_m in $\mathcal{W} \cap SP(H(\mathcal{W})_n)$. By $\delta(\phi)$ we mean the sentence obtained from $PCD(\phi)$ by replacing each occurrence of PCD_m with δ_m. Put $\beta_n = (\forall x_0, \ldots, x_n) \bigvee_{i<j\leq n} x_i \approx x_j$. We claim that $\mathcal{L}$ is finitely axiomatized relative to $\mathcal{W}$ by $\gamma(\delta_{n(n+1)/2}) \wedge \delta(\beta_n) \wedge \delta(\phi)$. □

Corollary. *Every quasivariety having pseudo-complemented congruence lattices and contained in a finitely generated quasivariety, is contained in a finitely axiomatizable, locally finite quasivariety.*

Corollary. *Let $\mathcal{K}$ be a finite set of finite algebras such that $SP(\mathcal{K})$ has pseudo-complemented congruence lattices. If $HS(\mathcal{K}) \subseteq SP(\mathcal{K})$ then $SP(\mathcal{K})$ is finitely axiomatizable.*

Proof. We can assume that $\mathcal{K} = HS(\mathcal{K})$ and each member of $\mathcal{K}$ has size at most n. Now we can axiomatize $\mathcal{K}$ by a positive universal sentence ϕ. Since $\mathcal{K}$ has Willard terms, there exists a finitely axiomatizable quasivariety $\mathcal{W}$ with Willard terms containing $\mathcal{K}$. Then $SP(\mathcal{K}) = \mathcal{W} \cap SP(H(\mathcal{W})_n) \cap SP(\mathrm{Mod}(\phi))$ and it is finitely axiomatizable. □

Corollary (R. Willard)**.** *Every congruence meet semi-distributive variety with finite residual bound is finitely axiomatizable.*

Definition. A quasivariety $\mathcal{K}$ has the *weak extension property* if for all $\mathbf{A} \in \mathcal{K}$ and for all $\alpha, \beta \in \operatorname{Con} \mathbf{A}$

$$\alpha \wedge \beta = 0 \rightarrow \alpha' \wedge \beta' = 0.$$

where $'$ is the *extension map* defined as $\alpha' = \bigcap\{\gamma \in \operatorname{Con}_{\mathcal{K}} \mathbf{A} : \alpha \leq \gamma\}$.

Theorem. *Let $\mathcal{W}$ be a quasivariety with pseudo-complemented congruence lattices, n be a positive integer, and ϕ be a universal sentence. Then $\mathcal{L} = \mathcal{W} \cap SP(H(\mathcal{W})_n) \cap SP(\operatorname{Mod}(\phi))$ is finitely axiomatizable relative to $\mathcal{W}$, provided there exists some quasivariety $\mathcal{E}$ with the weak extension property such that $\mathcal{L} \subseteq \mathcal{E} \subseteq SP(\operatorname{Mod}(\phi))$.*

Corollary. *Every finitely generated quasivariety with pseudo-complemented congruence lattices and the weak extension principle is finitely axiomatizable.*

Corollary (D. Pigozzi)**.** *Every finitely generated relatively congruence distributive quasivariety is finitely axiomatizable.*